International Journal of Theoretical Physics, Vol. 42, No. 1, January 2003 (© 2003)

Trefoil Symmetry V: Class Representations for the Minimal Clover Extension

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Received May 23, 2002

We construct four different real representations of the minimal vector clover extension and their covariant derivatives. We determine their relation to the basic representation and initiate the exploration of constrained superfields.

KEY WORDS: supersymmetry; superspace; graded symmetries; noncommutative field theory; representation theory.

1. INTRODUCTION

The introduction of the *Trefoil* symmetries and their *clover* extensions (Wills-Toro, 2001a,b; Wills-Toro *et al.*, 2001) has opened the way for constructing models with gradings beyond supersymmetry. The introduction of the *basic* superfield representation of the minimal vector clover extension has been successfully accomplished (Wills-Toro *et al.*, 2003). The study of covariantly constrained multiplets is greatly simplified with the introduction of novel representations. The real, the chiral, and the antichiral representations of supersymmetry are an eloquent example of the advantages of such novel representations for model building (Ferrara *et al.*, 1974). We are going to develop here four novel real representations. Chiral and antichiral representations will be discussed elsewhere.

2. CLASS (j) REPRESENTATION OF THE ACTION OF GENERATORS

In quite analogous way as we obtain chiral representations in supersymmetry, we ask for further representations of the minimal vector clover extension in

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superspace (Wills-Toro *et al.*, 2003). We can considerer novel combinations of the transformations

$$G(\chi,\xi,\bar{\xi},\beta)=e^{\mathrm{i}\chi_{(0)}^{-\mu}P_{(0)\mu}+\sum_{j=1,2,3}\left(\mathrm{i}\xi_{(j)}^{-r}T_{(j)r}+\mathrm{i}\bar{T}_{(j)\dot{r}}\bar{\xi}_{(j)}^{-\dot{r}}+\mathrm{i}\beta_{(j)}^{-\mu}P_{(j)\mu}\right)}. \tag{2.1}$$

where again, $(\chi_{(0)}^{-\mu},\xi_{(j)}^{-r},\bar{\xi}_{(j)}^{-\dot{r}},\beta_{(j)}^{-\mu})$ are the coordinates of the enhanced superspace (Wills-Toro *et al.*, 2003). We consider for instance the following three novel representations:

$$G_1(\chi,\xi,\bar{\xi},\beta)=G\big(\chi,\big(\xi_{(1)},0,0\big),\big(\bar{\xi}_{(1)},0,0\big),\big(0,\beta_{(2)},\beta_{(3)}\big) \tag{2.2}$$

$$\times\, G\big(0,\big(0,\xi_{(2)},\xi_{(3)}\big),\big(0,\bar{\xi}_{(2)},\bar{\xi}_{(3)}\big),\big(\beta_{(1)},0,0\big)\big), \tag{2.3}$$

$$G_2(\chi,\xi,\bar{\xi},\beta)=G\big(\chi,\big(0,\xi_{(2)},0\big),\big(0,\bar{\xi}_{(2)},0\big),\big(\beta_{(1)},0,\beta_{(3)}\big)\big) \tag{2.4}$$

$$\times\, G\big(0,\big(\xi_{(1)},0,\xi_{(3)}\big),\big(\bar{\xi}_{(1)},0,\bar{\xi}_{(3)}\big),\big(0,\beta_{(2)},0\big)\big), \tag{2.5}$$

$$G_3(\chi,\xi,\bar{\xi},\beta)=G\big(\chi,\big(0,0,\xi_{(3)}\big),\big(0,0,\bar{\xi}_{(3)}\big),\big(\beta_{(1)},\beta_{(2)},0\big)\big) \tag{2.6}$$

$$\times\, G\big(0,\big(\xi_{(1)},\xi_{(2)},0\big),\big(\bar{\xi}_{(1)},\bar{\xi}_{(2)},0\big),\big(0,0,\beta_{(3)}\big)\big). \tag{2.7}$$

From the products

$$G_1(a,\rho,\bar{\rho},\alpha)G_1(\chi,\xi,\bar{\xi},\beta)=G_1(\chi',\xi',\bar{\xi}',\beta'), \tag{2.8}$$

$$G_2(a,\rho,\bar{\rho},\alpha)G_2(\chi,\xi,\bar{\xi},\beta)=G_2(\chi'',\xi'',\bar{\xi}'',\beta''), \tag{2.9}$$

$$G_3(a,\rho,\bar{\rho},\alpha)G_3(\chi,\xi,\bar{\xi},\beta)=G_3(\chi''',\xi''',\bar{\xi}''',\beta'''). \tag{2.10}$$

We obtain for the G_1 case:

$$\beta_{(j)}^{\prime-\mu}=\beta_{(j)}^{-\mu}+\alpha_{(j)}^{-\mu},\quad j=1,2,3 \tag{2.11}$$

$$\xi_{(1)}^{\prime-r}=\xi_{(1)}^{-r}+\rho_{(1)}^{-r}+\frac{\mathrm{i}}{2}\beta_{(2)}^{-\rho}\alpha_{(3)}^{-\sigma}\eta^{r}(3,2)_{\sigma\rho}+\frac{\mathrm{i}}{2}\beta_{(3)}^{-\rho}\alpha_{(2)}^{-\sigma}\eta^{r}(2,3)_{\sigma\rho}, \tag{2.12}$$

$$\bar{\xi}_{(1)}^{\prime-\dot{r}}=\bar{\xi}_{(1)}^{-\dot{r}}+\bar{\rho}_{(1)}^{-\dot{r}}+\frac{\mathrm{i}}{2}\beta_{(2)}^{-\rho}\alpha_{(3)}^{-\sigma}\hat{\eta}^{\dot{r}}(3,2)_{\sigma\rho}+\frac{\mathrm{i}}{2}\beta_{(3)}^{-\rho}\alpha_{(2)}^{-\sigma}\hat{\eta}^{\dot{r}}(2,3)_{\sigma\rho}, \tag{2.13}$$

$$\xi_{(2)}^{\prime-r}=\xi_{(2)}^{-r}+\rho_{(2)}^{-r}+\mathrm{i}\beta_{(3)}^{-\rho}\alpha_{(1)}^{-\sigma}\eta^{r}(1,3)_{\sigma\rho}, \tag{2.14}$$

$$\bar{\xi}_{(2)}^{\prime-\dot{r}}=\bar{\xi}_{(2)}^{-\dot{r}}+\bar{\rho}_{(2)}^{-\dot{r}}+\mathrm{i}\beta_{(3)}^{-\rho}\alpha_{(1)}^{-\sigma}\hat{\eta}^{\dot{r}}(1,3)_{\sigma\rho}, \tag{2.15}$$

$$\xi_{(3)}^{\prime-r}=\xi_{(3)}^{-r}+\rho_{(3)}^{-r}+\mathrm{i}\beta_{(2)}^{-\rho}\alpha_{(1)}^{-\sigma}\eta^{r}(1,2)_{\sigma\rho}, \tag{2.16}$$

$$\bar{\xi}_{(3)}^{\prime-\dot{r}}=\bar{\xi}_{(3)}^{-\dot{r}}+\bar{\rho}_{(3)}^{-\dot{r}}+\mathrm{i}\beta_{(2)}^{-\rho}\alpha_{(1)}^{-\sigma}\hat{\eta}^{\dot{r}}(1,2)_{\sigma\rho}, \tag{2.17}$$

$$\chi_{(0)}^{\prime-\mu}=\chi_{(0)}^{-\mu}+a_{(0)}^{-\mu}$$

$$+\mathrm{i}\beta_{(2)}^{-\nu}\Big(\rho_{(2)}^{-r}+\frac{\mathrm{i}}{2}\beta_{(3)}^{-\rho}\alpha_{(1)}^{-\sigma}\eta^{r}(1,3)_{\sigma\rho}\Big)K_{r}(2)_{\nu}{}^{\mu}$$

$$+\,\mathrm{i}\beta_{(3)}^{-\nu}\big(\rho_{(3)}^{-r}+\frac{\mathrm{i}}{2}\beta_{(2)}^{-\rho}\alpha_{(1)}^{-\sigma}\eta^{r}(1,2)_{\sigma\rho}\big)K_{r}(3)_{\nu}{}^{\mu}$$
$$+\,\mathrm{i}\beta_{(2)}^{-\nu}\big(\bar{\rho}_{(2)}^{-\dot r}+\frac{\mathrm{i}}{2}\beta_{(3)}^{-\rho}\alpha_{(1)}^{-\sigma}\hat{\eta}^{\dot r}(1,3)_{\sigma\rho}\big)\hat{K}_{\dot r}(2)_{\nu}{}^{\mu}$$
$$+\,\mathrm{i}\beta_{(3)}^{-\nu}\big(\bar{\rho}_{(3)}^{-\dot r}+\frac{\mathrm{i}}{2}\beta_{(2)}^{-\rho}\alpha_{(1)}^{-\sigma}\hat{\eta}^{\dot r}(1,2)_{\sigma\rho}\big)\hat{K}_{\dot r}(3)_{\nu}{}^{\mu}$$
$$-\,\mathrm{i}\alpha_{(1)}^{-\nu}\big\{\xi_{(1)}^{-r}K_{r}(1)_{\nu}{}^{\mu}+\bar{\xi}_{(1)}^{-\dot r}\hat{K}_{\dot r}(1)_{\nu}{}^{\mu}\big\}. \tag{2.18}$$

The analogous result for $(\chi'', \xi'', \bar{\xi}'', \beta'')$ and $(\chi''', \xi''', \bar{\xi}''', \beta''')$ can be easily inferred. The transformation $(\chi, \xi, \bar{\xi}, \beta) \mapsto (\chi', \xi', \bar{\xi}', \beta')$ provides the class (1) differential representation of the action of operators:

$$\delta_{1P_{(0)\mu}} = \partial_{\chi_{(0)}^{-\mu}}, \tag{2.19}$$
$$\delta_{1T_{(1)r}} = \partial_{\xi_{(1)}^{-r}}, \tag{2.20}$$
$$\delta_{1T_{(2)r}} = \partial_{\xi_{(2)}^{-r}} - \mathrm{i}\beta_{(2)}^{-\nu}\hat{K}_{r}^{*}(2)_{\nu}{}^{\mu}\partial_{\chi_{(0)}^{-\mu}}, \tag{2.21}$$
$$\delta_{1T_{(3)r}} = \partial_{\xi_{(3)}^{-r}} - \mathrm{i}\beta_{(3)}^{-\nu}\hat{K}_{r}^{*}(3)_{\nu}{}^{\mu}\partial_{\chi_{(0)}^{-\mu}}, \tag{2.22}$$
$$\delta_{1\bar{T}_{(1)\dot r}} = \partial_{\bar{\xi}_{(1)}^{-\dot r}}, \tag{2.23}$$
$$\delta_{1\bar{T}_{(2)\dot r}} = \partial_{\bar{\xi}_{(2)}^{-\dot r}} - \mathrm{i}\beta_{(2)}^{-\nu}K_{\dot r}^{*}(2)_{\nu}{}^{\mu}\partial_{\chi_{(0)}^{-\mu}}, \tag{2.24}$$
$$\delta_{1\bar{T}_{(3)\dot r}} = \partial_{\bar{\xi}_{(3)}^{-\dot r}} - \mathrm{i}\beta_{(3)}^{-\nu}K_{\dot r}^{*}(3)_{\nu}{}^{\mu}\partial_{\chi_{(0)}^{-\mu}}, \tag{2.25}$$

$$\delta_{1P_{(1)\sigma}} = \partial_{\beta_{(1)}^{-\sigma}} - \mathrm{i}\beta_{(3)}^{-\rho}\eta^{r}(3,1)_{\rho\sigma}\partial_{\xi_{(2)}^{-r}} - \mathrm{i}\beta_{(2)}^{-\rho}\eta^{r}(2,1)_{\rho\sigma}\partial_{\xi_{(3)}^{-r}}$$
$$-\,\mathrm{i}\beta_{(3)}^{-\rho}\hat{\eta}^{\dot r}(3,1)_{\rho\sigma}\partial_{\bar{\xi}_{(2)}^{-\dot r}} - i\beta_{(2)}^{-\rho}\hat{\eta}^{\dot r}(2,1)_{\rho\sigma}\partial_{\bar{\xi}_{(2)}^{-\dot r}}$$
$$-\,\frac{1}{2}\beta_{(3)}^{-\rho}\beta_{(2)}^{-\nu}\big(\eta^{r}(3,1)_{\rho\sigma}\hat{K}_{r}^{*}(2)_{\nu}{}^{\mu}+\hat{\eta}^{\dot r}(3,1)_{\rho\sigma}K_{\dot r}^{*}(2)_{\nu}{}^{\mu}\big)\partial_{\chi_{(0)}^{-\mu}}$$
$$-\,\frac{1}{2}\beta_{(2)}^{-\rho}\beta_{(3)}^{-\nu}\big(\eta^{r}(2,1)_{\rho\sigma}\hat{K}_{r}^{*}(3)_{\nu}{}^{\mu}+\hat{\eta}^{\dot r}(2,1)_{\rho\sigma}K_{\dot r}^{*}(3)_{\nu}{}^{\mu}\big)\partial_{\chi_{(0)}^{-\mu}}$$
$$-\,\mathrm{i}\big(\xi_{(1)}^{-r}K_{r}(1)_{\sigma}{}^{\mu}+\bar{\xi}_{(1)}^{-\dot r}\hat{K}_{\dot r}(1)_{\sigma}{}^{\mu}\big)\partial_{\chi_{(0)}^{-\mu}}, \tag{2.26}$$
$$\delta_{1P_{(2)\sigma}} = \partial_{\beta_{(2)}^{-\sigma}} - \frac{\mathrm{i}}{2}\beta_{(3)}^{-\rho}\big(\eta^{r}(3,2)_{\rho\sigma}\partial_{\xi_{(1)}^{-r}}+\hat{\eta}^{\dot r}(3,2)_{\rho\sigma}\partial_{\bar{\xi}_{(1)}^{-\dot r}}\big), \tag{2.27}$$
$$\delta_{1P_{(3)\sigma}} = \partial_{\beta_{(3)}^{-\sigma}} - \frac{\mathrm{i}}{2}\beta_{(2)}^{-\rho}\big(\eta^{r}(2,3)_{\rho\sigma}\partial_{\xi_{(1)}^{-r}}+\hat{\eta}^{\dot r}(2,3)_{\rho\sigma}\partial_{\bar{\xi}_{(1)}^{-\dot r}}\big), \tag{2.28}$$

Analogous results are inferred for the class (2) representation $\delta_{2\mathcal{O}}$ and class (3) representation $\delta_{3\mathcal{O}}$ for each generator $\mathcal{O}$ of the minimal vector clover extension.

Strictly speaking, we should write $\delta_{\mathcal{O}}^{\Phi_1}$ instead of $\delta_{1\mathcal{O}}$ in the equations above, since the differential representation considered refers to some fixed superfield representation Φ_1.

Since the basic representation (Wills-Toro *et al.*, 2003) and the class (1), class (2), and class (3) ones handle symmetrically the parameters ξ and $\bar{\xi}$ (associated to the antisymmetric vector generators), we might understand these representations as real. The class (1), class (2), and class (3) representations concern a novelty of the trefoil symmetries: besides the right–left or chiral–antichiral representations, we are dealing with "class" representations.

It is also a long but straightforward calculation to verify that the given representation (2.19)–(2.28) verify the relations

$$\begin{aligned}\left[\!\left[\delta_{1\,P_{(k)\sigma}}, \delta_{1\,P_{(j)\alpha}}\right]\!\right] &= -\mathrm{i}(\eta^{r}(k,\,j)_{\sigma\alpha}\delta_{1\,T_{(k\dagger j)r}} \\ &\quad + \hat{\eta}^{\dot{r}}(k,\,j)_{\sigma\alpha}\delta_{1\,\bar{T}_{(k\dagger j)\dot{r}}}); \quad k \neq j\end{aligned} \tag{2.29}$$

$$\left[\!\left[\delta_{1\,T_{(j)s}}, \delta_{1\,P_{(k)\sigma}}\right]\!\right] = -\mathrm{i}\delta_{jk}K_{s}(k)_{\sigma}^{\mu}\delta_{1\,P_{(0)\mu}}, \tag{2.30}$$

$$\left[\!\left[\delta_{1\,\bar{T}_{(j)\dot{s}}}, \delta_{1\,P_{(k)\sigma}}\right]\!\right] = -\mathrm{i}\delta_{jk}\hat{K}_{\dot{s}}(k)_{\sigma}^{\mu}\delta_{1\,P_{(0)\mu}}, \tag{2.31}$$

and all further q-commutation relations between the given class (1) differential representations vanish.

3. CLASS (j) REPRESENTATIONS OF THE COVARIANT DERIVATIVES

The covariant derivatives for the class (j) representations will be given explicitly for the class (1) case. The further classes are easily inferred.

The covariant derivatives $D_{P_{(1)\sigma}}$, $D_{T_{(2)r}}$, $D_{\bar{T}_{(2)\dot{r}}}$, $D_{T_{(3)}r}$, $D_{\bar{T}_{(3)\dot{r}}}$ of the basic representation q-commute with each other (Wills-Toro *et al.*, 2003). This suggests that superfields can be given that vanish under the action of all these covariant derivatives. The class (1) representation provides an appropriate basis for the description of such covariantly constrained superfields as follows from its covariant derivatives:

$$D_{1\,T_{(1)r}} = \partial_{\xi_{(1)}^{-r}} + \mathrm{i}\beta_{(1)}^{-\nu}\,\hat{K}_{r}^{*}(1)_{\nu}{}^{\mu}\partial_{\chi_{(0)}^{-\mu}}, \tag{3.1}$$

$$D_{1\,\bar{T}_{(1)\dot{r}}} = \partial_{\bar{\xi}_{(1)}^{-\dot{r}}} + \mathrm{i}\beta_{(1)}^{-\nu}\,K_{\dot{r}}^{*}(1)_{\nu}{}^{\mu}\partial_{\chi_{(0)}^{-\mu}}, \tag{3.2}$$

$$D_{1\,T_{(2)r}} = \partial_{\xi_{(2)}^{-r}}, \qquad D_{1\,\bar{T}_{(2)\dot{r}}} = \partial_{\bar{\xi}_{(2)}^{-\dot{r}}}, \tag{3.3}$$

$$D_{1\,T_{(3)r}} = \partial_{\xi_{(3)}^{-r}}, \qquad D_{1\,\bar{T}_{(3)\dot{r}}} = \partial_{\bar{\xi}_{(3)}^{-\dot{r}}}, \tag{3.4}$$

$$D_{1P_{(1)\sigma}} = \partial_{\beta_{(1)}^{-\sigma}}, \tag{3.5}$$

$$
\begin{aligned}
D_{1P_{(2)\sigma}} = {} & \partial_{\beta_{(2)}^{-\sigma}} + \mathrm{i}\beta_{(1)}^{-\rho}\left\{\eta^{r}(1,2)_{\rho\sigma}\partial_{\xi_{(3)}^{-r}} + \hat{\eta}^{\dot{r}}(1,2)_{\rho\sigma}\partial_{\bar{\xi}_{(3)}^{-\dot{r}}}\right\} \\
& + \frac{\mathrm{i}}{2}\beta_{(3)}^{-\rho}\left\{\eta^{r}(3,2)_{\rho\sigma}\partial_{\xi_{(1)}^{-r}} + \hat{\eta}^{\dot{r}}(3,2)_{\rho\sigma}\partial_{\bar{\xi}_{(1)}^{-\dot{r}}}\right\} \\
& + \mathrm{i}\left(\xi_{(2)}^{-r}K_{r}(2)_{\sigma}{}^{\mu} + \bar{\xi}_{(2)}^{-\dot{r}}\hat{K}_{\dot{r}}(2)_{\sigma}{}^{\mu}\right)\partial_{\chi_{(0)}^{-\mu}}, \qquad (3.6)
\end{aligned}
$$

$$
\begin{aligned}
D_{1P_{(3)\sigma}} = {} & \partial_{\beta_{(3)}^{-\sigma}} + \mathrm{i}\beta_{(1)}^{-\rho}\left\{\eta^{r}(1,3)_{\rho\sigma}\partial_{\xi_{(2)}^{-r}} + \hat{\eta}^{\dot{r}}(1,3)_{\rho\sigma}\partial_{\bar{\xi}_{(2)}^{-\dot{r}}}\right\} \\
& + \frac{\mathrm{i}}{2}\beta_{(2)}^{-\rho}\left\{\eta^{r}(2,3)_{\rho\sigma}\partial_{\xi_{(1)}^{-r}} + \hat{\eta}^{\dot{r}}(2,3)_{\rho\sigma}\partial_{\bar{\xi}_{(1)}^{-\dot{r}}}\right\} \\
& + \mathrm{i}\left(\xi_{(3)}^{-r}K_{r}(3)_{\sigma}{}^{\mu} + \bar{\xi}_{(3)}^{-\dot{r}}\hat{K}_{\dot{r}}(3)_{\sigma}{}^{\mu}\right)\partial_{\chi_{(0)}^{-\mu}}. \qquad (3.7)
\end{aligned}
$$

The simple structure of the mutually q-commuting covariant derivatives $D_{1P_{(1)\sigma}}$, $D_{1T_{(2)r}}$, $D_{1\bar{T}_{(2)\dot{r}}}$, $D_{1T_{(3)r}}$, $D_{1\bar{T}_{(3)\dot{r}}}$ implies that a covariantly constrained class (1) superfield can be constructed that does not depend on the parameters $\beta_{(1)}$, $\xi_{(2)}$, $\bar{\xi}_{(2)}$, $\xi_{(3)}$, $\bar{\xi}_{(3)}$.

A long but straightforward computation proves that the following relations hold:

$$
\begin{aligned}
\left[\!\left[D_{1P_{(k)\sigma}}, D_{1P_{(j)\alpha}}\right]\!\right] &= \mathrm{i}\left(\eta^{r}(k,j)_{\sigma\alpha}D_{1T_{(k\dagger j)r}}\right. \\
&\qquad \left. + \hat{\eta}^{\dot{r}}(k,j)_{\sigma\alpha}D_{1\bar{T}_{(k\dagger j)\dot{r}}}\right), \quad k \neq j \qquad (3.8)
\end{aligned}
$$

$$
\left[\!\left[D_{1T_{(k)s}}, D_{1P_{(j)\sigma}}\right]\!\right] = \mathrm{i}\delta_{jk}K_{s}(k)_{\sigma}{}^{\mu}\partial_{\chi_{(0)}^{-\mu}}, \qquad (3.9)
$$

$$
\left[\!\left[D_{1\bar{T}_{(k)\dot{s}}}, D_{1P_{(j)\sigma}}\right]\!\right] = \mathrm{i}\delta_{jk}\hat{K}_{\dot{s}}(k)_{\sigma}{}^{\mu}\partial_{\chi_{(0)}^{-\mu}}, \qquad (3.10)
$$

and all further q-commutations among these covariant derivatives vanish. The covariant derivatives (3.1)–(3.7) q-commute with all the class (1) representation of generators $\delta_{1\mathcal{O}}$ given in (2.19)–(2.28), as expected. Accordingly, the basic, the class (1), class (2), and class (3) representations share the same main features, and the algebraic relations are maintained by covariance. In particular, the algebraic relations among covariant derivatives obtained for the basic representation hold in all the novel representations.

4. RELATIONS AMONG THE BASIC AND CLASS (*j*) REPRESENTATIONS

For the construction of models with the considered symmetries it is useful to recognize the relation between the diverse representations. In particular, we want to determine a relation among the diverse superfield representations in terms of "shifts" in the enhanced superspace.

We will present the relation among the basic representation and the class (1) representation of a superfield. The relation between the basic and the class (2) and class (3) representations is easily inferred. The relation between two different representations, say class (j) and class (k), is obtained by relating the class (j) to the basic representation, and then relating the basic and class (k) representation.

Let $\delta_{\mathcal{O}}$ be a differential operator in the basic representation and let $\Phi(\chi, \xi, \bar{\xi}, \beta)$ be a superfield also in the basic representation. Let $\delta_{i\mathcal{O}}$ and $\Phi_i(\chi, \xi, \bar{\xi}, \beta)$ be, respectively, the operator and superfield in the class (i) representation. They are related by a differential operator S_i, so that

$$\Phi(\chi, \xi, \bar{\xi}, \beta) = e^{S_i} \Phi_i(\chi, \xi, \bar{\xi}, \beta), \tag{4.1}$$

$$\delta_{\mathcal{O}} = e^{S_i} \delta_{i\mathcal{O}} e^{-S_i}. \tag{4.2}$$

Accordingly

$$\delta_{\mathcal{O}} \Phi(\chi, \xi, \bar{\xi}, \beta) = e^{S_i} \delta_{i\mathcal{O}} \Phi_i(\chi, \xi, \bar{\xi}, \beta). \tag{4.3}$$

The operator S_i has trivial index assignment, hence the relation (4.2) can be expanded in orders of S_i:

$$\delta_{i\mathcal{O}} = e^{-S_i} \delta_{\mathcal{O}} e^{S_i} = \delta_{\mathcal{O}} + [\delta_{\mathcal{O}}, S_i] + \frac{1}{2}[[\delta_{\mathcal{O}}, S_i], S_i] + \cdots \tag{4.4}$$

where the expansion should be pursued until all further terms are proved to be vanishing.

In particular, for the relation between the basic and the class (1) representation we find

$$\begin{aligned} S_1 = {} & \frac{\mathrm{i}}{2}\left(\xi_{(1)}^{-r} \beta_{(1)}^{-\nu} \hat{K}_r^*(1)_\nu{}^\mu + \bar{\xi}_{(1)}^{-\dot{r}} \beta_{(1)}^{-\nu} K_{\dot{r}}^*(1)_\nu{}^\mu\right) \partial_{\chi_{(0)}^{-\mu}} \\ & + \frac{\mathrm{i}}{2}\left(\xi_{(2)}^{-r} \beta_{(2)}^{-\nu} \hat{K}_r^*(2)_\nu{}^\mu + \bar{\xi}_{(2)}^{-\dot{r}} \beta_{(2)}^{-\nu} K_{\dot{r}}^*(2)_\nu{}^\mu\right) \partial_{\chi^{-\mu}{}_{(0)}} \\ & - \frac{\mathrm{i}}{2}\left(\xi_{(3)}^{-r} \beta_{(3)}^{-\nu} \hat{K}_r^*(3)_\nu{}^\mu + \bar{\xi}_{(1)}^{-\dot{r}} \beta_{(3)}^{-\nu} K_{\dot{r}}^*(3)_\nu{}^\mu\right) \partial_{\chi^{-\mu}{}_{(0)}} \\ & - \frac{\mathrm{i}}{2} \beta_{(1)}^{-\sigma} \sum_{i \neq 1} \beta_{(1\dagger i)}^{-\rho} \left\{\eta^r(1\dagger i, 1)_{\rho\sigma} \partial_{\xi_{(i)}}^{-r} + \hat{\eta}^{\dot{r}}(1\dagger i, 1)_{\rho\sigma} \partial_{\bar{\xi}_{(i)}}^{-\dot{r}}\right\} \\ & - \frac{1}{24} \beta_{(1)}^{-\sigma} \sum_{i \neq 1} \beta_{(1\dagger i)}^{-\rho} \beta_{(i)}^{-\nu} \left\{\eta^r(1\dagger i, 1)_{\rho\sigma} \hat{K}_r^*(i)_\nu{}^\mu \right. \\ & \left. + \hat{\eta}^{\dot{r}}(1\dagger i, 1)_{\rho\sigma} K_{\dot{r}}^*(i)_\nu{}^\mu\right\} \partial_{\chi_{(0)}}^{-\mu}. \end{aligned} \tag{4.5}$$

The form of S_2 and S_3 is easily inferred.

The reader might verify that the term $\frac{1}{2}[[\delta_{\mathcal{O}}, S_i], S_i]$ in (4.4) will be relevant for establishing the relation between $\delta_{\mathcal{O}}$ and $\delta_{i\mathcal{O}}$ or between $D_{\mathcal{O}}$ and $D_{i\mathcal{O}}$. But no further terms contribute in expansion (4.4) due to the particular form of the S_i.

The action of e^{S_1} on Φ_1 produces a "shift" in the enhanced superspace. The relation between the superfields $\Phi(\chi, \xi, \overline{\xi}, \beta)$ and $\Phi_1(\chi, \xi, \overline{\xi}, \beta)$—the former in the basic and the later in the class (1) representation—is given by

$$\begin{aligned}
&\Phi(\chi, \xi, \overline{\xi}, \beta) \\
&= \Phi_1 \Bigg(\chi_{(0)}^{-\mu} + \frac{\mathrm{i}}{2}\big(\xi_{(1)}^{-r}\beta_{(1)}^{-\nu}\hat{K}_r^*(1)_\nu{}^\mu + \overline{\xi}_{(1)}^{-\dot{r}}\beta_{(1)}^{-\nu}K_{\dot{r}}^*(1)_\nu{}^\mu\big) \\
&- \frac{\mathrm{i}}{2}\big(\xi_{(2)}^{-r}\beta_{(2)}^{-\nu}\hat{K}_r^*(2)_\nu{}^\mu + \overline{\xi}_{(2)}^{-\dot{r}}\beta_{(2)}^{-\nu}K_{\dot{r}}^*(2)_\nu{}^\mu\big) \\
&- \frac{\mathrm{i}}{2}\big(\xi_{(3)}^{-r}\beta_{(3)}^{-\nu}\hat{K}_r^*(3)_\nu{}^\mu + \overline{\xi}_{(3)}^{-\dot{r}}\beta_{(3)}^{-\nu}K_{\dot{r}}^*(3)_\nu{}^\mu\big) \\
&- \frac{1}{24}\beta_{(1)}^{-\sigma}\sum_{i\neq 1}\beta_{(1\dagger i)}^{-\rho}\beta_{(i)}^{-\nu}\big\{\eta^r(1\dagger i, 1)_{\rho\sigma}\hat{K}_r^*(i)_\nu{}^\mu + \hat{\eta}^{\dot{r}}(1\dagger i, 1)_{\rho\sigma}K_{\dot{r}}^*(i)_\nu{}^\mu\big\}, \\
&\left(\xi_{(1)}^{-r}, \xi_{(2)}^{-r} - \frac{\mathrm{i}}{2}\beta_{(1)}^{-\sigma}\beta_{(3)}^{-\rho}\eta^r(3,1)_{\rho\sigma}, \xi_{(3)}^{-r} - \frac{\mathrm{i}}{2}\beta_{(1)}^{-\sigma}\beta_{(2)}^{-\rho}\eta^r(2,1)_{\rho\sigma}\right), \\
&\left(\xi_{(1)}^{-\dot{r}}, \overline{\xi}_{(2)}^{-\dot{r}} - \frac{\mathrm{i}}{2}\beta_{(1)}^{-\sigma}\beta_{(3)}^{-\rho}\hat{\eta}^{\dot{r}}(3,1)_{\rho\sigma}, \overline{\xi}_{(3)}^{-\dot{r}} - \frac{\mathrm{i}}{2}\beta_{(1)}^{-\sigma}\beta_{(2)}^{-\rho}\hat{\eta}^{\dot{r}}(2,1)_{\rho\sigma}\right), \\
&\big(\beta_{(1)}, \beta_{(2)}, \beta_{(3)}\big)\Bigg)
\end{aligned} \tag{4.6}$$

The relation between Φ and Φ_2 and Φ_3 is easily inferred.

Let A_1 be a superfield in class (1) representation fulfilling the covariant constraints:

$$D_{1P_{(1)\sigma}}A_1 = 0, \tag{4.7}$$

$$D_{1T_{(2)r}}A_1 = 0, \qquad D_{1\bar{T}_{(2)\dot{r}}}A_1 = 0, \tag{4.8}$$

$$D_{1T_{(3)r}}A_1 = 0, \qquad D_{1\bar{T}_{(3)\dot{r}}}A_1 = 0. \tag{4.9}$$

Such a constrained superfield will be called a *class (1) superfield*. Hence

$$A_1 = A_1\left(\chi_{(0)}^{-\mu}, \big(\xi_{(1)}, 0, 0\big), \big(\overline{\xi}_{(1)}, 0, 0\big), \big(0, \beta_{(1)}, \beta_{(3)}\big)\right). \tag{4.10}$$

Correspondingly, its relation to its basic representation A_{1b} is given by

$$\begin{aligned}
&A_{1b}(\chi, \xi, \overline{\xi}, \beta) \\
&= A_1\Bigg(\chi_{(0)}^{-\mu} + \frac{\mathrm{i}}{2}\big(\xi_{(1)}^{-r}\beta_{(1)}^{-\nu}\hat{K}_r^*(1)_\nu{}^\mu + \overline{\xi}_{(1)}^{-\dot{r}}\beta_{(1)}^{-\nu}K_{\dot{r}}^*(1)_\nu{}^\mu\big)
\end{aligned}$$

$$
\begin{aligned}
&- \frac{\mathrm{i}}{2}\big(\xi_{(2)}^{-r}\beta_{(2)}^{-\nu}\hat{K}_r^*(2)_\nu{}^\mu + \bar{\xi}_{(2)}^{-\dot{r}}\beta_{(2)}^{-\nu}K_{\dot{r}}^*(2)_\nu{}^\mu\big)\\
&- \frac{\mathrm{i}}{2}\big(\xi_{(3)}^{-r}\beta_{(3)}^{-\nu}\hat{K}_r^*(3)_\nu{}^\mu + \bar{\xi}_{(3)}^{-\dot{r}}\beta_{(3)}^{-\nu}K_{\dot{r}}^*(3)_\nu{}^\mu\big)\\
&- \frac{1}{24}\beta_{(1)}^{-\sigma}\sum_{i\neq 1}\beta_{(1\dagger i)}^{-\rho}\beta_{(i)}^{-\nu}\big\{\eta^r(1\dagger i,1)_{\rho\sigma}\hat{K}_r^*(i)_\nu{}^\mu + \hat{\eta}^{\dot{r}}(1\dagger i,1)_{\rho\sigma}K_{\dot{r}}^*(i)_\nu{}^\mu\big\},\\
&\big(\xi_{(1)}^{-r},0,0),(\bar{\xi}_{(1)}^{-\dot{r}},0,0),(0,\beta_{(2)}^{-\sigma},\beta_{(3)}^{-\sigma})\Big),
\end{aligned}
\tag{4.11}
$$

which corresponds in this case to a purely space-time shifting. The corresponding relations for class (2) and class (3) (constrained) superfields are easily inferred.

5. CLASS (0) REPRESENTATION OF THE ACTION OF GENERATORS

The basic (Wills-Toro *et al.*, 2003), the class (1), class (2), and class (3) representations of the minimal vector clover extension obtained so far share the property of being real representations. There is nevertheless a further real representation that merits attention. The covariant derivatives $D_{T_{(1)r}}, D_{\bar{T}_{(1)\dot{r}}}, D_{T_{(2)r}}, D_{\bar{T}_{(2)\dot{r}}}, D_{T_{(3)r}}, D_{\bar{T}_{(3)\dot{r}}}$ q-commute among them, hence we except to have a further real representation in which all this operators are derivatives with respect to the variables $\xi_{(j)}$ and $\bar{\xi}_{(j)}$. Consider then the following representation

$$
\begin{aligned}
G_0(\chi,\xi,\bar{\xi},\beta) = G\big(\chi,(0,0,0),(0,0,0),\big(\beta_{(1)},\beta_{(2)},\beta_{(3)}\big)\big)\\
\times G\big(0,\big(\xi_{(1)},\xi_{(2)},\xi_{(3)}\big),\big(\bar{\xi}_{(1)},\bar{\xi}_{(2)},\bar{\xi}_{(3)}\big),(0,0,0)\big).
\end{aligned}
\tag{5.1}
$$

The composition of two such transformations

$$
G_0(a,\rho,\bar{\rho},\alpha)G_0(\chi,\xi,\bar{\xi},\beta) = G_0(\breve{\chi},\breve{\xi},\breve{\bar{\xi}},\breve{\beta}),
\tag{5.2}
$$

leads to the superspace coordinates

$$
\begin{aligned}
\breve{\chi}_{(0)}^{-\mu} = {}& \chi_{(0)}^{-\mu} + a_{(0)}^{-\mu} + \mathrm{i}\beta_{(j)}^{-\nu}\big(\rho_{(j)}^{-r}K_r(j)_\nu{}^\mu + \bar{\rho}_{(j)}^{-\dot{r}}\hat{K}_{\dot{r}}(j)_\nu{}^\mu\big)\\
&- \frac{1}{3}\big(\beta_{(j)}^{-\nu} - \alpha_{(j)}^{-\nu}\big)\sum_{k\neq j}\beta_{(k\dagger j)}^{-\rho}\alpha_{(k)}^{-\sigma}\big\{\eta^r(k,k\dagger j)_{\sigma\rho}K_r(j)_\nu{}^\mu\\
&+ \hat{\eta}^{\dot{r}}(k,k\dagger j)_{\sigma\rho}\hat{K}_{\dot{r}}(j)_\nu{}^\mu\big\},
\end{aligned}
\tag{5.3}
$$

$$
\breve{\xi}_{(j)}^{-r} = \xi_{(j)}^{-r} + \rho_{(j)}^{-r} + \frac{\mathrm{i}}{2}\sum_{k\neq j}\beta_{(k)}^{-\rho}\alpha_{(k\dagger j)}^{-\sigma}\eta^r(k\dagger j,k)_{\sigma\rho},
\tag{5.4}
$$

$$
\breve{\bar{\xi}}_{(j)}^{-\dot{r}} = \bar{\xi}_{(j)}^{-\dot{r}} + \bar{\rho}_{(j)}^{-\dot{r}} + \frac{\mathrm{i}}{2}\sum_{k\neq j}\beta_{(k)}^{-\rho}\alpha_{(k\dagger j)}^{-\sigma}\hat{\eta}^{\dot{r}}(k\dagger j,k)_{\sigma\rho},
\tag{5.5}
$$

$$\breve{\beta}^{-\mu}_{(j)} = \beta^{-\mu}_{(j)} + \alpha^{-\mu}_{(j)}. \tag{5.6}$$

From the transformation $(\chi, \xi, \bar{\xi}, \beta) \mapsto (\breve{\chi}, \breve{\xi}, \breve{\bar{\xi}}, \breve{\beta})$ we infer what will be called the class (0) representation of the generators:

$$\delta_{0P_{(0)\mu}} = \partial_{\chi^{-\mu}_{(0)}}, \tag{5.7}$$

$$\delta_{0T_{(j)r}} = \partial_{\xi^{-r}_{(j)}} - \mathrm{i}\beta^{-\nu}_{(j)} \hat{K}^*_r(j)_\nu{}^\mu \partial_{\chi^{-\mu}_{(0)}}, \tag{5.8}$$

$$\delta_{0\bar{T}_{(j)\dot{r}}} = \partial_{\bar{\xi}^{-\dot{r}}_{(j)}} - \mathrm{i}\beta^{-\nu}_{(j)} K^*_{\dot{r}}(j)_\nu{}^\mu \partial_{\chi^{-\mu}_{(0)}}, \tag{5.9}$$

$$\begin{aligned}\delta_{0P_{(k)\sigma}} = \partial_{\beta^{-\sigma}_{(k)}} &- \frac{\mathrm{i}}{2}\sum_{j\neq k} \beta^{-\rho}_{(k\dagger j)}\left\{\eta^r(k\dagger j, k)_{\rho\sigma}\partial_{\xi^{-r}_{(j)}} + \hat{\eta}^{\dot{r}}(k\dagger j, k)_{\rho\sigma}\partial_{\bar{\xi}^{-\dot{r}}_{(j)}}\right\} \\ &- \frac{1}{3}\sum_{j\neq k} \beta^{-\rho}_{(k\dagger j)}\beta^{-\nu}_{(j)}\left\{\eta^r(k\dagger j, k)_{\rho\sigma}\hat{K}^*_r(i)_\nu{}^\mu \right. \\ &\left. + \hat{\eta}^{\dot{r}}(k\dagger j, k)_{\rho\sigma} K^*_{\dot{r}}(j)_\nu{}^\mu\right\}\partial_{\chi^{-\mu}_{(0)}}.\end{aligned} \tag{5.10}$$

It is straightforward to verify that they fulfil the q-commutators relations:

$$[\![\delta_{0P_{(k)\sigma}}, \delta_{0P_{(j)\alpha}}]\!] = -\mathrm{i}\left(\eta^r(k, j)_{\sigma\alpha}\delta_{0T_{(k\dagger j)r}} + \hat{\eta}^{\dot{r}}(k, j)_{\sigma\alpha}\delta_{0\bar{T}_{(k\dagger j)\dot{r}}}\right), \tag{5.11}$$

$$[\![\delta_{0T_{(j)s}}, \delta_{0P_{(k)\sigma}}]\!] = -\mathrm{i}\delta_{jk} K_s(k)_\sigma{}^\mu \delta_{0P_{(0)\mu}}, \tag{5.12}$$

$$[\![\delta_{0\bar{T}_{(j)\dot{s}}}, \delta_{0P_{(k)\sigma}}]\!] = -\mathrm{i}\delta_{jk} \hat{K}_{\dot{s}}(k)_\sigma{}^\mu \delta_{0P_{(0)\mu}}. \tag{5.13}$$

All further q-commutation relations among such differential representations vanish.

6. CLASS (0) REPRESENTATIONS OF THE COVARIANT DERIVATIVES

As already anticipated, the covariant derivatives in the class (0) representation adopt a particularly simple form for the covariant derivatives $D_{0T(j)_r}$ and $D_{0\bar{T}(j)_{\dot{r}}}$:

$$D_{0T(j)r} = \partial_{\xi^{-r}_{(j)}}, \tag{6.1}$$

$$D_{0\bar{T}(j)\dot{r}} = \partial_{\bar{\xi}^{-\dot{r}}_{(j)}}, \tag{6.2}$$

$$\begin{aligned}D_{0P_{(k)\sigma}} = \partial_{\beta^{-\sigma}_{(k)}} &+ \frac{\mathrm{i}}{2}\sum_{j\neq k} \beta^{-\rho}_{(k\dagger j)}\left\{\eta^r(k\dagger j, k)_{\rho\sigma}\partial_{\xi^{-r}_{(j)}} + \hat{\eta}^{\dot{r}}(k\dagger j, k)_{\rho\sigma}\partial_{\bar{\xi}^{-\dot{r}}_{(j)}}\right\} \\ &+ i\left\{\xi^{-r}_{(k)} K_r(k)_\sigma{}^\mu + \bar{\xi}^{-\dot{r}}_{(k)}\hat{K}_{\dot{r}}(k)_\sigma{}^\mu\right\}\partial_{\chi^{-\mu}_{(0)}}\end{aligned}$$

$$- \frac{1}{3} \sum_{j \neq k} \beta^{-\rho}_{(k\dagger j)} \beta^{-\nu}_{(j)} \{ \eta^{r}(k\dagger j, k)_{\rho\sigma} \hat{K}^{*}_{r}(i)_{\nu}{}^{\mu}$$

$$+ \hat{\eta}^{\dot{r}}(k\dagger j, k)_{\rho\sigma} K^{*}_{\dot{r}}(j)_{\nu}{}^{\mu} \} \partial_{\chi^{-\mu}_{(0)}}. \tag{6.3}$$

Again, they fulfill the expected commutation relations

$$[\![D_{0P_{(k)\sigma}}, D_{0P_{(j)\alpha}}]\!] = +\mathrm{i}\big(\eta^{r}(k,i)_{\sigma\alpha} D_{0T_{(k\dagger j)r}} + \hat{\eta}^{\dot{r}}(k,i)_{\sigma\alpha} D_{0\bar{T}_{(k\dagger j)\dot{r}}}\big), \quad j \neq k \tag{6.4}$$

$$[\![D_{0T_{(j)s}}, D_{0P_{(k)\sigma}}]\!] = +\mathrm{i}\delta_{jk} K_{s}(k)_{\sigma}{}^{\mu} \partial_{\chi^{-\mu}_{(0)}}, \tag{6.5}$$

$$[\![D_{0\bar{T}_{(j)\dot{s}}}, D_{0P_{(k)\sigma}}]\!] = +\mathrm{i}\delta_{jk} \hat{K}_{\dot{s}}(k)_{\sigma}{}^{\mu} \partial_{\chi^{-\mu}_{(0)}}. \tag{6.6}$$

and all further q-commutations among these covariant derivatives vanish. The covariant derivatives (6.1)–(6.3) q-commute with all the class (0) representations of the algebra generators (5.7)–(5.10) as expected.

7. RELATION AMONG THE BASIC AND THE CLASS (0) REPRESENTATION

Let $\delta_{\mathcal{O}}$ be a differential operator and $\Phi(\chi, \xi, \bar{\xi}, \beta)$ a superfield both in the basic representation. Let $\delta_{0\mathcal{O}}$ and $\Phi_0(\chi, \xi, \bar{\xi}, \beta)$ be the same operator and superfield but in the class (0) representation. They are related by a differential operator S_0 with trivial index assignment:

$$\Phi(\chi, \xi, \bar{\xi}, \beta) = e^{S_0} \Phi_i(\chi, \xi, \bar{\xi}, \beta), \tag{7.1}$$

$$\delta_{\mathcal{O}} = e^{S_0} \delta_{0\mathcal{O}} e^{-S_0}. \tag{7.2}$$

Accordingly

$$\delta_{\mathcal{O}} \Phi(\chi, \xi, \bar{\xi}, \beta) = e^{S_0} \delta_{0\mathcal{O}} \Phi_i(\chi, \xi, \bar{\xi}, \beta) \tag{7.3}$$

The differential operator S_0 has the form

$$S_0 = -\frac{\mathrm{i}}{2} \sum_{j=1,2,3} \{ \xi^{-r}_{(j)} \hat{K}^{*}_{r}(j)_{\nu}{}^{\mu} + \bar{\xi}^{-\dot{r}}_{(j)} K^{*}_{\dot{r}}(j)_{\nu}{}^{\mu} \} \beta^{-\nu}_{(j)} \partial_{\chi^{-\mu}_{(0)}}. \tag{7.4}$$

Accordingly, the action of e^{S_0} on Φ_0 produces a space-time shift in the enhanced superspace:

$$\Phi(\chi, \xi, \bar{\xi}, \beta) =$$

$$\Phi_0 \left(\chi^{-\mu}_{(0)} - \frac{\mathrm{i}}{2} \sum_{j=1,2,3} \{ \xi^{-r}_{(j)} \hat{K}^{*}_{r}(j)_{\nu}{}^{\mu} + \bar{\xi}^{-\dot{r}}_{(j)} K^{*}_{\dot{r}}(j)_{\nu}{}^{\mu} \} \beta^{-\nu}_{(j)}, \xi, \bar{\xi}, \beta \right). \tag{7.5}$$

Let A_0 be a superfield in class (0) representation fulfilling the covariant constraints:

$$D_{0T(j)_r} A_0 = 0, \qquad D_{0\bar{T}(j)_{\dot{r}}} A_0 = 0. \tag{7.6}$$

Such a constrained superfield will be called a *class* (0) *superfield*. Hence

$$A_0 = A_0\big(\chi_{(0)}^{-\mu}, (0, 0, 0), (0, 0, 0), \big(\beta_{(1)}, \beta_{(2)}, \beta_{(3)}\big)\big). \tag{7.7}$$

Its relation to its basic representation A_{0b} will be given by

$$A_{0b}(\chi, \xi, \bar{\xi}, \beta) = A_0\left(\chi_{(0)}^{-\mu} - \frac{\mathrm{i}}{2}\sum_{j=1,2,3}\left\{\xi_{(j)}^{-r}\hat{K}_r^*(j)_\nu{}^\mu + \bar{\xi}_{(j)}^{-\dot{r}}K_{\dot{r}}^*(j)_\nu{}^\mu\right\}\beta_{(j)}^{-\nu}, 0, 0, \beta\right). \tag{7.8}$$

8. CONCLUSIONS

We have obtained four novel real representations of the minimal vector clover extension. The relation among them has also been obtained. The novel constrained superfields will offer clover multiplets with particular field components content. These representations will prove very useful in relating to well-established models in Quantum Field Theory. Further representations in the spirit of chiral and antichiral representations will be explored elsewhere.

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